Fusion-Reflection

Self-Supervised Learning

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Abstract

By analyzing learning from the perspective of *knowledge acquisition*, a number of common limitations are overcome. Modeling efficacy is proposed as an empirical measure of knowledge, providing a concrete, mathematical means of "acquiring knowledge" via gradient ascent. A specific network architecture is described, a hierarchical analog of node-labeled Hidden Markov Models, and its evaluation and learning laws are derived. In empirical studies using a hand-printed character recognition task, an unsupervised network was able to discover n-gram statistics from groups of letter images, and to use these statistics to enhance its ability to later identify individual letters.

Introduction

In the pursuit of synthetic intelligence, the study of learning is of central importance. Many tasks which seem simple to us in fact require a vast hierarchy of knowledge. Consider the task of identifying a pictured animal: this first requires an implicit understanding of more fundamental concepts such as visual continuity, edges, orientation, depth, form, and the nature of identity itself—just to name a few. Not only does this represent a vast amount of knowledge, but our own understanding of some of the crucial concepts is still incomplete. Ideally, the study of learning will produce an *adaptive* intelligence that can discover these principles on its own, and perhaps even reveal them to us, rather than vice-versa.

For an intelligent system to be capable of learning, aspects of its behavior must be modifiable. We can imagine the system as a black box with input, output, and set of modifiable, internal variables, or *free parameters*. These internal parameters determine the behavior of the box and can be envisioned as a set of controlling dials. Inside the box is some fixed algorithm, the *evaluation law*, which combines the input with the state of the dials to generate output. It is presumed that this algorithm is sufficiently robust that *some* configuration of the dials will produce the desired behavior. The goal of the *learning law* is to find this configuration.

Consider, for instance, the name-that-animal problem. Upon unpacking our newly arrived ACME All-Purpose Black Box, we find its dials in a random configuration. After hooking a scanner to the box's input and a printer to its output, we observe the expected: the box does no better than chance. In fact, it does worse, outputting "blblblblbl"—which is not an animal at all. How do we determine the configuration of dials that will make the box name-that-animal?

The most direct approach is simply to compute the proper configuration, and set the dials. Doing this, however, requires an analytic solution to the particular evaluation law in question. Suppose, for example, we wish the box to light a bulb when it sees the word "idea". If the box had four dials with settings from "a" to "z", we could simply dial in i-d-e-a, and we would be done. On the other hand, if the evaluation law processed the input through some complex equation involving the dial settings, then there may be no straightforward, one-step method of determining a good configuration. In practice, the most flexible evaluation laws fall into the latter category, having no known analytic solutions.

When the solution is not analytically attainable, we can attempt to incrementally improve performance instead. This basic technique is known as *hill climbing* or *gradient ascent*, for reasons which the following example should make clear: Consider a two-dial box that we wish to configure to perform some task, such as lighting a bulb when it sees a cow. For any particular configuration of the dials, the box will have some measurable performance. In this case, the average brightness of the bulb when it sees a cow minus that when it doesn't would serve as a good measure of performance—it is least when the box is performing badly, and greatest when, and only when, the box is performing perfectly. These three values together, the two dial settings and the measure of performance, can be used to define a topological map, where the dials define the x,y location and the corresponding performance defines the altitude. Finding the optimal configuration amounts to locating the highest peak on the map. For all but the most trivial problems, an exhaustive search—computing the performance at every point on the map—is computationally out of the question. Instead, we can start at a random location and, by computing the direction of slope (the gradient), simply travel uphill until we reach a peak. This is much like hiking to the top of an unexplored mountain while blindfolded—the best we can do is to walk uphill and hope that the first peak we find is the highest one (or at least high enough to meet our needs). The same technique applies regardless of the number of free parameters (dials), though the hill analogy is difficult to visualize in the higher dimensional spaces.

In order to perform gradient ascent, we need a concrete measure of the system's performance; the most direct method is to compare its output against *desired* output. For instance, in the name-that-animal problem, the natural measure of performance would be the percentage of correctly named animals. *Supervised* algorithms[1]—algorithms which learn from examples of correct answers—are generally measured this way, by comparing their answers against the given examples. Gradient ascent using this measure of performance means finding those incremental changes to the free parameters that will bring the system's output ever closer to the desired output.

For many real-world problems, however, this kind of performance measure is insufficient as a learning guide. The assumption is made, when hill climbing, that there will be a slope to follow, that at each step there will be some incremental change of the parameters that can improve performance. What if the problem is so difficult that this is not true? Consider again the problem of lighting a bulb when the input image represents a cow. From an initial random configuration, the system has no concept of spatial continuity, edges, or orientation—let alone of cow-ness. A great deal of knowledge must be acquired *before* the system can even *begin* to improve its cow-identifying ability. If improving that ability is its only guide to learning, then it has no guidance in acquiring the prerequisite knowledge. Returning to the hill climbing analogy: as the problem becomes more difficult, the performance surface looks less and less like the smooth hill we would like to imagine, and begins to look more like Pinnacles National Park—a vast plane with narrow spikes jutting up here and there. Imagine trying to find the highest peak *there* while blindfolded.

As a possible way around this problem, instead of measuring the final output performance, we could measure the performance at each successive stage of abstraction. For instance, rather than having a single black box solve the cow-light problem in one big step, we could cascade a number of boxes end-to-end, the first receiving the picture and the last lighting the bulb. Then we might train the first to abstract edges from the picture, the next to abstract curves from the edges, then shape from the curves, form from the shape, and finally cow-ness from the form. Our hope would be that each of these smaller problems would be simple enough to solve with gradient ascent. Unfortunately, predetermining the various stages of abstraction presumes considerable knowledge of the final solution. If we had that, we probably wouldn't need an adaptive algorithm at all.

[1] The most commonly used (and in many ways archetypal) supervised algorithm is Back Propagation. See [Rumelhart, Hinton, and Williams (1986)].

Instead of predetermining the stages of abstraction, we could supply an algorithm to discover these stages on its own. Many *unsupervised* learning laws[2]—which do not receive output examples—use an algorithmic measure of performance to abstract "useful" features from their input. Imagine a series of boxes, each ideally containing the same algorithm, and each learning to perform a single stage of abstraction on its input. The algorithm may, for example, form categories based on groupings of similar input patterns and convey those categories to the next stage. Using this approach, a box receiving images of handwritten characters might form a unique category for each letter of the alphabet and pass these categorized letters to the next box, which might form a unique category for each word, and so on.

What kind of final output could we expect from such a system, where each stage of abstraction is guided only by a *local* measure of performance? It is hard to say. Having eliminated the global measure of performance in favor of the more easily solved local measures, the system is left with no overall goal. Each stage abstracts certain features and sends them onward, without ever receiving feedback as to how *globally* valuable those features are. They may be useless, and more importantly, crucial features may be ignored altogether. For instance, the box categorizing letters may not realize that O and Q need to be differentiated, since they are visually similar. Conversely, it may form multiple categories for a letter like A, which can be drawn in more than one way. Resolving ambiguities like this without feedback is impossible—the necessary information is simply unavailable[3] .

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To advance learning beyond these limitations, we need a *global* measure of the process of abstraction—a measure of the system's knowledge as a whole. When we considered only the final output performance, we ignored the earlier stages of abstraction; on the other hand, when we considered only a single stage at a time, we ignored the need to coordinate abstraction from a global perspective. Measuring the system's *total* knowledge would solve both of these problems. Even the earliest stages of abstraction represent knowledge, so a system using knowledge as its measure of performance has guidance from the ground up. Likewise, such a system must coordinate abstraction from a global perspective, since (as in the letters example above) certain knowledge may not become available until higher-level knowledge has been learned. In terms of gradient ascent, using knowledge as our performance measure should define a gradient topology that is easily traversed from beginning to end by normal hill climbing.

But how do we measure "knowledge"? This might seem a difficult problem, but fortunately, an *empirical* measure of knowledge exists in the *system's ability to model its input*—that is, its ability to generate patterns similar in nature to the input patterns. Specifically, if the system could, from scratch, generate "imaginary" examples of input patterns with exactly the same probability with which those patterns would actually occur, then the system can be said to have a complete working knowledge of the input. Consider a black box whose input is a camera which takes random snapshots of everyday life on earth. If that box could in turn fabricate imaginary pictures of life on earth in exactly the same probabilistic distribution as they would actually occur, then it must contain all relevant knowledge about life on earth. Every picture would implicitly convey the laws of physics—ropes would hang in catenaries, objects would rest on supporting surfaces, and frisbees would frequently be underneath cars. Every human emotion would be understood—the child walking a fence would look amused, while the woman watching her child would look distraught. It would be safe to deduce that every aspect of visible existence must *in some way* be

[2] See [Becker, S. (1991)] for more about unsupervised algorithms.

[3] "Feedback" , in this case, refers to any information received from *other* areas of the system, and need not necessarily come from "higher" levels of abstraction. See [Becker, S. (1992)].

encoded in that box, whether by brute force as a catalog of all possible pictures, or by abstraction as the fundamental laws of nature.

The key point is that a model does not record and playback like a VCR; it learns, then imagines things it has never seen before. And unlike the previously described systems, which produced as output some sort of analysis of a specific input, a model generates patterns from scratch—from the *knowledge* it has acquired about the general nature of its input.

Notice that the processes of observing (taking input) and acting (producing output) have become separate functions. We have shifted the focus from training the system to do a particular thing—performance enhancement—to having it simply learn about its input—knowledge acquisition. As such, the metaphor is no longer a black box with input and output, but rather a black box with input only. This input is, in essence, the box's window to the world—everything it needs to know is available in the input. The "desired output" of a supervised system becomes simply more input to the model—learning that the caption "cow" goes with the picture of a cow is no different from learning that *this* kind of leg belongs on a sheep, and *that* kind of leg belongs on a table.

In fact, since our goal is to acquire knowledge, we need not generate "output" patterns at all. Modeling efficacy is merely our measure of knowledge, *not* the ultimate goal. A model's output neither represents any transformation of the input, nor is it the desired end—it is simply proof of the model's knowledge. Applying that knowledge (as in pattern classification, pattern completion, etc...) is a separate problem and can be dealt with at a later time. (Fortunately, the structure required to obtain knowledge by modeling is generally useful in applying that knowledge as well. An example of this is provided in the specific treatment of *Furf-I*, the algorithm presented later in this paper.)

As an illustration of the modeling approach, consider the basic two-input Boolean logic functions (AND, OR, XOR, etc...). Traditionally, these are viewed as mapping functions, taking two inputs and producing a single output. From that perspective, the following tables are definitive:

AND

x	y	z
F	F	F
F	T	F
T	F	F
T	T	T

OR

x	y	z
F	F	F
F	T	T
T	F	T
T	T	T

XOR

x	y	z
F	F	F
F	T	T
T	F	T
T	T	F

These tables specify, for a given function, how to map a given input (x,y) to its corresponding output (z). A model, on the other hand, makes no distinction between input and output. Instead, once trained, it generates each possible aggregate pattern (x,y,z) with some defined probability:

AND

x	y	z	P
F	F	F	30%
F	T	F	20%
T	F	F	15%
T	T	F	0%
F	F	T	0%
F	T	T	0%
T	F	T	0%
T	T	T	35%

OR

x	y	z	P
F	F	F	30%
F	T	F	0%
T	F	F	0%
T	T	F	0%
F	F	T	0%
F	T	T	20%
T	F	T	15%
T	T	T	35%

XOR

x	y	z	P
F	F	F	30%
F	T	F	0%
T	F	F	0%
T	T	F	35%
F	F	T	0%
F	T	T	20%
T	F	T	15%
T	T	T	0%

Notice that these tables, although not directly specifying an input-output mapping, do contain all the necessary information to perform such a mapping. Further, these tables contain information about the relative frequencies of the patterns—information which the mapping tables could not convey. For example, if the OR model generates a pattern with y,z both *true*, then we can compute a 64% probability that x is also *true*. (Performing these computations does not necessarily require an exhaustive look-up table as we have here. See the *Furf-I* evaluation laws, below.)

Designing a Model

In order to use modeling efficacy as our measure of system performance, the system must first be a stochastic pattern generator. It is here, in designing the method of pattern generation, that we set the limits on the model's ability to learn: If a model cannot easily generate a particular pattern, then it cannot easily learn that pattern, since learning in this case *is* "learning to generate". Since the model is generating new patterns from scratch, its methods must be stochastic, or it would always generate the same thing. For example, consider a system that models biased coin tosses. Each time we ask the system to generate a pattern, it should *randomly* select heads or tails according to the bias of the coin being modeled. In this way, the system will behave just like the coin, turning up heads sometimes, tails others. Or, imagine a system that models snowflakes. The free parameters here could define the branching probabilities in a fractal algorithm—each newly generated snowflake would potentially be unique.

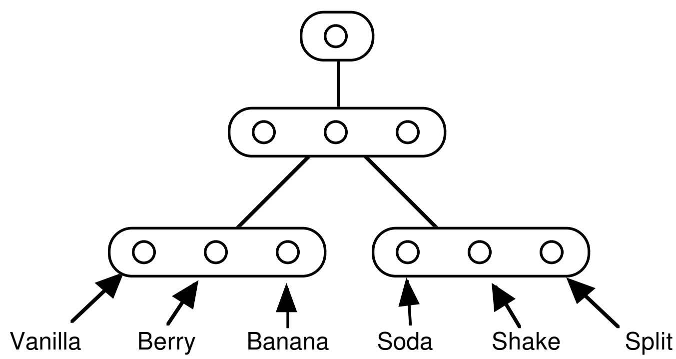
Training an adaptive system requires finding the gradient of the performance measure. In this case, the performance measure is modeling efficacy, which, in practical terms, can be measured as the probability that the model would have generated the training set (the set of patterns the model is given to learn from). Consider the biased-coin model again. To create a training set, we could flip the real coin four times, which might, for example, produce H-H-T-H. The probability of our model generating that training set is $B^3(1-B)$, where B is the model's bias for turning up heads. The gradient of this function is $3B^2-4B^3$, which is positive when $B<.75$, negative when $B>.75$, and zero when $B=.75$. So, as we would expect, the gradient leads us to choose a bias of .75, which best represents the training set[4] .

The next section describes a particular modeling architecture, *Furf-I*. The following sections derive the corresponding evaluation and learning laws. It is important to remember that these derivations are based on the method of pattern *generation*—so a firm understanding of the generation algorithm is an absolute prerequisite to understanding the evaluation and learning laws which follow.

[4] There are difficulties in using small training sets such as the one given in the example. See the *Small Training Sets* sub-section in the *Future Work* section of this paper for a brief analysis.

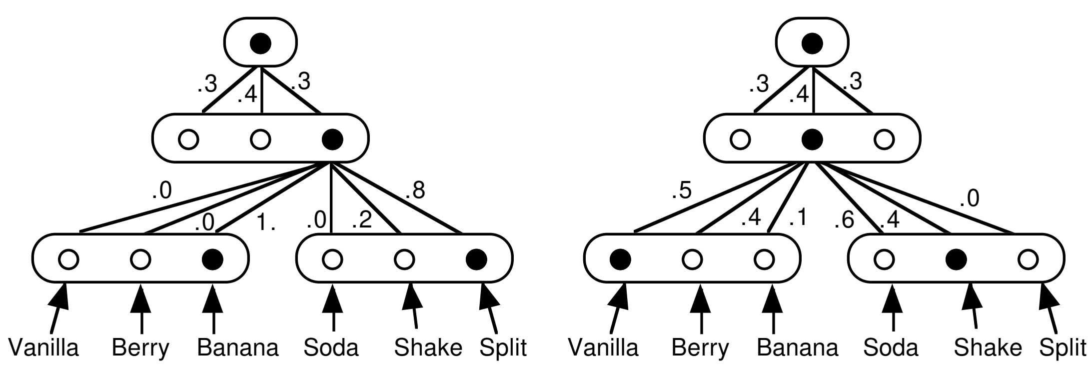
Furf Nets

A Furf-I network is organized as a hierarchy of groups of nodes, where each node belongs to one and only one group. Connections can be made between any two nodes from different groups. Typically, however, connections are made at the group level, where all nodes in one group are fully interconnected to all nodes in another group. In the following illustrations, a connection shown between two groups indicates full nodal interconnection:



Dessert Network

The network operates as a stochastic pattern generator from the top down. Pattern generation always begins by selecting the single node in the topmost group. Each connection downward from a selected node to a node below dictates the probability of selecting that lower node. Using these probabilities, a single node from each group below is randomly selected. Since only one node is selected from each group, the nodes within a group represent mutually exclusive possibilities. This process continues downward from each newly selected node until the bottom is reached. The resulting state of the bottom layer defines the generated pattern:



"Banana Split" **"Vanilla Shake"**

The *evaluation law* for a Furf network is a two-pass algorithm consisting of an *upward* and a *downward* pass[5]. The upward evaluation law computes the probability that any particular node would generate a given pattern. For instance, from the dessert network example above, the upward evaluation law could tell us the probability that the middle node, of the middle group,

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[5] This *upward/downward* evaluation law is very similar to the *forward/backward* algorithm used in Hidden Markov Models [Baum, L. E. (1972)]. In fact, a node-labeled HMM can be constructed from a two-group, recurrent Furf network (see *Temporal Data* in the *Future Work* section of this paper)

would generate "Vanilla Shake" (in this case, 20%). Since the top-most node originates all generated patterns, the *overall* probability of the network generating a given pattern can be computed by applying the upward evaluation law to that top-most node. For example, the dessert network would, as a whole, generate "Vanilla Shake" with at least an 8% probability (or more, depending on the probabilities associated with the left-most node in the middle group).

The *downward evaluation law* computes the probability that any particular node was selected during the generation of a given pattern. This is useful for both pattern completion and categorization. For example, we can present the partial pattern "? Split" to the dessert network above by setting the Vanilla, Berry, and Banana nodes as equally likely, while setting the Split node to 1, and the Soda and Shake nodes to 0. After upward/downward evaluation, the Vanilla and Berry nodes would have a probability of 0, while the Banana node would have a probability of 100%, thus completing the pattern as "Banana Split". In general, however, the completed pattern will not be so definite. The pattern "? Shake", for instance, would produce the probabilities that a randomly selected shake would be Vanilla, Berry, and Banana—each of which is non-zero in this example.

There are many ways to *categorize* an input using the upward/downward evaluation law. The simplest way is to use pattern completion to fill in the category—such as filling in the caption of a picture. Another way is to use a group higher up in the network as output, assuming that the nodes within that group represent the categories we are interested in. The character recognition network described later is an example of this approach.

The *learning law* is derived in terms of pattern-generating performance—it attempts to maximize the probability that the Furf net would generate the training patterns. Because learning maximizes a *global* measure of performance, every parameter in the Furf net can influence any other. The flow of information during a learning cycle is from the bottom up, and then back down again. In essence, the low level concepts are fused into higher and higher level concepts, and then this high level information is reflected back downward to help fine-tune the lower-level concept formation. This full cycle allows all concepts to potentially interact, while still maintaining a computationally local learning law.

The next sections detail the derivations of these three laws—the upward and downward evaluation laws and the learning law. The mathematics may appear somewhat complex at first, but this is largely due to the lack of an attractive representation for probabilistic relationships. Most of the derivations are fairly straightforward given a good understanding of Bayes theorem, and the results are pleasantly simple if one can see past the probabilistic notation. A brief review of probabilistic mathematics is given for those who aren't familiar with the subject. The final mathematical forms of the evaluation and learning laws are summarized at the beginning of the *empirical studies* section.

Stochastic Mathematics

This section is a review of probabilistic mathematics. There is nothing unique to Furf networks in this section, so it can be safely skipped by those already familiar with the subject.

Basic Notation and Universal Equalities

The central concept of probabilistic mathematics is expressed by

$P(A)$ – the probability of A

There are a number of equivalent ways to view the meaning of P(A). Within the context of this paper, the following provides the most useful interpretation: Imagine a collection of samples—say a box of Lego building blocks. Each sample will have specific attributes—a chosen block may be blue and have six pegs, for instance. If a sample is chosen at random, P(A) is simply the probability that the sample will have attribute A. E.g., P(blue) is the probability that a randomly chosen block will be blue. If there are a finite number of samples, this probability is easily computed by counting:

$$P(A) = \frac{N_A}{N_{total}}$$ – P(A) is the fraction of samples with attribute A

Here, N_A is the number of samples with attribute A, and N_{total} is the total number of samples. By definition, a sample either has a given attribute or it doesn't. Thus, where $P(\overline{A})$ is the probability that a sample will *not* have attribute A,

$$P(A) + P(\overline{A}) = 1$$

Any logical combination of attributes can be treated as a single cumulative attribute. For instance, P(blue,six pegs) is the probability that a randomly chosen block will be blue and have six pegs. Similarly, P(blue *or* six pegs) is the probability that it will be blue *or* have six pegs. Obviously the order in which the attributes are specified is not important:

$P(A,B) = P(B,A)$ – the probability of A and B

$P(A \text{ or } B) = P(B \text{ or } A)$ – the probability of A or B

P(A) tells us the *a priori* probability of A—the probability of A for a completely randomly chosen sample. If, however, the sample is chosen from some *subset* of the possible samples, then the probability that it will have attribute A may be very different from the a priori probability. For example, say all six-pegged blocks are blue, and all four-pegged blocks are red. Randomly choosing a block may turn up red only 50% of the time. This is the a priori probability of red. However, randomly choosing a four-pegged block will turn up red 100% of the time. This is the probability that a block is red *given* that it has four pegs. This relationship is expressed by the notation

$P(A|B)$ – the probability of A given B

Using the counting method, we can compute $P(A|B)$ as the ratio of samples with both A and B to those with just B:

$$P(A|B) = \frac{N_{AB}}{N_B} = \frac{P(A,B)}{P(B)}$$ – Bayes Theorem

Recalling that $P(A,B) = P(B,A)$, we can write:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$ – Bayes Theorem

Extending these equations for multiple attributes is straightforward:

$$P(A,B|C,D) = \frac{P(A,B,C,D)}{P(C,D)} = \frac{P(C,D|A,B)P(A,B)}{P(C,D)} = \frac{P(A,C,D|B)P(B)}{P(C|D)P(D)}, \text{ etc...}$$

Simplifying Assumptions

When we know the relationship between two or more attributes, we can generally substitute simpler equations in place of the more general ones. If the probability of A is independent of B, then the probability of finding both together is simply the product of their independent probabilities:

$$P(A|B) = P(A) \rightarrow P(A,B) = P(A)P(B)$$ – junction of independent attributes

If A and B are independent, then so are $\overline{A}$ and $\overline{B}$, thus:

$$P(A|B) = P(A) \rightarrow P(A \text{ or } B) = 1-(1-P(A))(1-P(B))$$

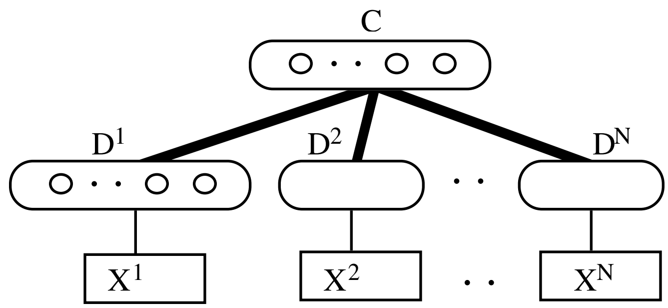
– union of independent attributes

If two events are mutually exclusive, then the probability of either one being true is just the sum of their independent probabilities:

$$P(A,B) = 0 \rightarrow P(A \text{ or } B) = P(A) + P(B)$$ – mutually exclusive attributes

Derivation of the Upward Evaluation Law

Given a node C_i and a input pattern X covered by C, the probability $P(X|C_i)$ can be recursively computed from the bottom up. To do this, X must be divided into smaller patterns X^1 through X^N according to coverage by C's descendants D^1 through D^N as shown:



Since events D_1^n through D_N^n are mutually exclusive:

$$P(X^n|C_i) = \sum_j P(D_j^n|C_i)P(X^n|D_j^n) \quad (1)$$

And since the groups D^1 though D^N are treated independently:

$$P(X|C_i) = \prod_n P(X^n|C_i) \quad (2)$$

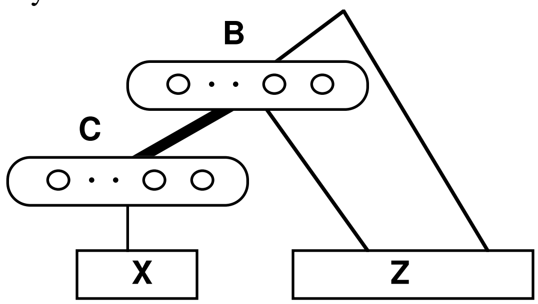
Combining (1) and (2) gives:

$$\boxed{P(X|C_i) = \prod_n \sum_j P(D_j^n|C_i)P(X^n|D_j^n)} \quad (3)$$

In practice, this value becomes very small within a couple of layers. It turns out, however, that it is only the relative ratios of these values within the group that matter during the downward pass, so it is acceptable to uniformly scale the entire group to keep these values within a manageable range.

Derivation of the Downward Evaluation Law

Given a pattern T generated by the model, it is possible to compute the probability that a particular node C_i was selected during the generation of T. This probability $P(C_i|T)$ can be computed recursively from the top down. To do this, the input pattern T must be divided into two parts, X and Z, according to coverage by C as shown:



The limitations imposed by the topology give:

(1) $$P(T|C_i) = P(X|C_i)P(Z|C_i) \;\rightarrow\; P(C_i|T) = \frac{P(X|C_i)P(C_i|Z)P(Z)}{P(T)}$$

(2) $$P(T|B_j) = P(X|B_j)P(Z|B_j) \;\rightarrow\; P(B_j|Z) = \frac{P(B_j|T)P(T)}{P(X|B_j)P(Z)}$$

(3) $$P(C_i|Z) = \sum_j P(B_j|Z)P(C_i|B_j)$$

Substituting (2) into (3) into (1) gives:

(4) $$\boxed{P(C_i|T) = P(X|C_i)\sum_j \frac{P(C_i|B_j)P(B_j|T)}{P(X|B_j)}}$$

The values $P(X|C_i)$ and $P(X|B_j)$ have already been computed during the upward pass (equations (3) and (1) in the Upward Evaluation Law Derivation, respectively).

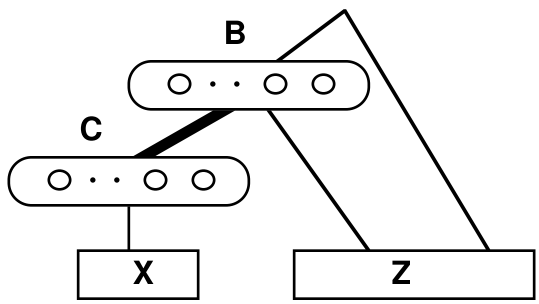
As a side note: If the parameters in the layers above C are approximately uniform (implying that learning has not yet progressed in the upper layers), then the backward pass has no effect other than to normalize the probabilities within the groups:

(5) $$P(C_m|B_j) \cong P(C_n|B_j) \text{ for all } j,m,n \;\rightarrow\; P(C_i|T) \cong \frac{P(X|C_i)}{\sum_n P(X|C_n)}$$

Derivation of the Furf-I Learning Law

Given a fixed topology, the only free parameters in a Furf-I network are the probabilities associated with the node-to-node connections. Since no analytical method for determining the optimal set of parameters is known, an incremental approach must be used. The gradient derived below is poorly behaved in practice in that its length increases by many orders of magnitude as the model learns. As an alternative to pure gradient ascent, a well-behaved variant is derived, which proves to be an incremental analog to the Baum-Welch algorithm used in training Hidden Markov Models [Baum, L. E. (1972)].

To compute the gradient, the derivative $\frac{\partial P(T)}{\partial P(C_i|B_j)}$ is needed for every pair of connected nodes B_j and C_i. Since we are dealing with a *set* of patterns (the "training set") instead of a single pattern, we will use a slightly different notation than before: T^a shall mean the a'th pattern in the pattern set T. As with the downward evaluation, each input pattern T^a must be divided into two parts $X^a \& Z^a$ according to coverage by C:



The probability $P(T)$ of the model generating the entire pattern set is simply the product of the probabilities of the model generating each of the individual patterns in the set. As such, the derivative of $P(T)$ with respect to any variable U can be rewritten in terms of the individual pattern probabilities as follows:

(1) $$P(T) = \prod_a P(T^a) \quad \rightarrow \quad \frac{\partial P(T)}{\partial U} = P(T) \sum_a \frac{\left(\frac{\partial P(T^a)}{\partial U}\right)}{P(T^a)}$$

The individual pattern probabilities can be written thus:

(2) $$P(T^a) = \sum_j P(B_j) P(T^a|B_j) = \sum_j \left[P(B_j) P(Z^a|B_j) \sum_i P(C_i|B_j) P(X^a|C_i) \right]$$

Differentiating produces:

(3) $$\frac{\partial P(T^a)}{\partial P(C_i|B_j)} = P(B_j) P(Z^a|B_j) P(X^a|C_i) = \frac{P(T^a) P(B_j|T^a) P(X^a|C_i)}{P(X^a|B_j)}$$

Combining (1) and (3) yields:

$$\frac{\partial P(T)}{\partial P(C_i|B_j)} = P(T)\sum_a \frac{P(B_j|T^a)P(X^a|C_i)}{P(X^a|B_j)} \tag{4}$$

If the parameters were unconstrained, the gradient defined by equation (4) could be followed directly. However, the set of probabilities descending from a node to a single group ($P(C_1|B_j)$ through $P(C_N|B_j)$) must sum to one. Stated geometrically, this set of probabilities can be viewed as a point in N_C dimensional space, constrained to lie on the hyperplane defined by:

$$\sum_{i=1}^{N_C} P(C_i|B_j) = 1 \tag{5}$$

Thus, the influence of the gradient must be limited to motion parallel to that hyperplane. The projection of the gradient defined by equation (4) onto the plane defined by equation (5) is:

$$\partial_{B_j,C_i} = P(T)\left[\sum_a \frac{P(B_j|T^a)P(X^a|C_i)}{P(X^a|B_j)} - \frac{1}{N_C}\sum_k \sum_a \frac{P(B_j|T^a)P(X^a|C_k)}{P(X^a|B_j)}\right] \tag{6}$$

Dropping $P(T)$ (as it is effectively constant throughout an epoch) and swapping summation order allows us to compute a delta vector at each pattern presentation rather than having to sum over the entire epoch:

$$\boxed{\partial^a_{B_j,C_i} = \frac{P(B_j|T^a)}{P(X^a|B_j)}\left[P(X^a|C_i) - \frac{1}{N_C}\sum_k P(X^a|C_k)\right]} \tag{7}$$

The update rule utilizing this projected gradient would then be:

$$P^{a+1}(C_i|B_j) = P^a(C_i|B_j) + R_{B_jC}\partial^a_{B_j,C_i} \tag{8}$$

where R_{B_jC} is a scalar learning rate associated with the connections from node B_j to group C. In practice, the variance of the $P(X^a|B_j)$ in the denominator should steadily increase as learning progresses. Unfortunately, this results in dramatic fluctuations in the length of the learning vector, particularly if $P(X^a|B_j)$ approaches zero for some X^a. If this gradient is to be followed directly, setting R_{B_jC} in inverse proportion to a decaying average of $|\partial^a_{B_j,C}|$ may compensate for the increasing variance of the per-pattern gradient over time.

Liberal application of Bayes theorem to equation (6) gives:

$$\partial_{B_j,C_i} = P(T)\left[\frac{1}{P(C_i|B_j)}\sum_a P(B_j\&C_i|T^a) - \frac{1}{N_C}\sum_k \frac{1}{P(C_k|B_j)}\sum_a P(B_j\&C_k|T^a)\right] \tag{9}$$

From here it is easy to find the conditions, subject to equation (5), under which the epoch gradient goes to zero:

$$|\partial_{B_j,C}| = 0 \quad \text{when} \quad P(C_i|B_j) = \frac{\sum_a P(B_j \& C_i|T^a)}{\sum_k \sum_a P(B_j \& C_k|T^a)} = \frac{\sum_a P(B_j \& C_i|T^a)}{\sum_a P(B_j|T^a)} \tag{10}$$

When equation (10) is satisfied, we have found a local maximum of $P(T)$. In the Baum-Welch re-estimation algorithm [Baum, L. E. (1972)], every parameter $P(C_i|B_j)$ is simply set according to equation (10) at the end of each training epoch. Changing these parameters alters future computations of (10), resulting in successively better approximations to the true convergence location with each epoch.

Waiting until the end of an epoch to update the parameters requires more storage than the incremental approach, and also requires the size of the epoch to be finite. Equation (10) can be converted to an incremental algorithm by observing that:

$$\frac{\sum_n F^n}{\sum_n G^n} \text{ is approached by V if } V^{n+1} = \left[1-G^n L\right]V^n + F^n L \text{ for small L} \tag{11}$$

Combining (10) and (11) into an incremental algorithm and expanding $P(B_j \& C_i|T^a)$ gives:

$$\boxed{P^{a+1}(C_i|B_j) = \left[1-P^a(B_j|T^a)L\right]P^a(C_i|B_j) + L\frac{P^a(B_j|T^a)P^a(X^a|C_i)P^a(C_i|B_j)}{P^a(X^a|B_j)}} \tag{12}$$

In this form, all expressions on the right hand side denote relatively local values that have already been computed during evaluation. The learning rate L should be a positive scalar less than one, as demonstrated by the following analysis:

If we define $R^a_{B_i} = P^a(B_j|T^a)L$, (12) can be written:

$$P^{a+1}(C_i|B_j) = \left[1-R^a_{B_j}\right]P^a(C_i|B_j) + R^a_{B_j}P^a(C_i|B_j \& X^a) \tag{13}$$

This shows that the plasticity of a connection is proportional to the target node’s activation. Since $P^a(B_j|T^a) \leq 1$, setting $L \leq 1$ implies $R^a_{B_j} \leq 1$, which makes equation (13) a non-negative weighted average of two probability vectors (over i). Since both probability vectors independently sum to one, their weighted average must also sum to one.

Empirical Studies

The Equations:

The Furf-I evaluation and learning laws can be summarized as follows:

$$V_{g_jf} = \sum_i w_{g_jf_i} U_{f_i}, \quad U_{g_j} = \prod_f V_{g_jf}, \quad D_{g_j} = U_{g_j} \sum_k \frac{w_{h_kg_j} D_{h_k}}{V_{h_kg}} \qquad \text{— Evaluation Law}$$

$$\Delta w_{g_jf_i} = -L w_{g_jf_i} D_{g_j} \left(1 - \frac{U_{f_i}}{V_{g_jf}} \right) \qquad \text{— Learning Law}$$

where f, g, and h are the input, current, and output groups, and i, j, and k are the respective input, current, and output node indices within each of those groups; *L* is the learning rate; w is a weight, or connection strength, between two nodes; V is a partial product; U is a node's upward activation; and D is a node's downward activation. Refer to the derivational sections for the probabilistic interpretations of these values.

The Task:

The primary empirical studies of the Furf-I algorithm used a hand-printed character recognition task. The data for this task consisted of about 9,000 hand-printed, lower-case letters and apostrophes from a single writer[6], rendered with anti-aliasing into two-bit deep, 12x12 images. No separation of training and testing data was made, but since the majority of the learning often happened within the first epoch, it was not a pressing issue (as pattern evaluation, which provided performance statistics, always preceded pattern training). The corpus was derived from written English text, and was presented in order, including the separating spaces (which brought the actual epoch size to nearly 12,000).

Context-Free Recognition:

The basic topology used for this task was a four-layer network, beginning with a 12x12(x4) input layer (144 four-node groups), through a 3x3(x18) middle layer, to a 28-node output group, and finally to the single-node top group[7]. Thus, each pixel of the 12x12 input image was represented by a *group* of four nodes, the pixel's shade (one of four values) determining which of the four nodes would have its *upward* activation set to 1 while the other three nodes' activations were set to 0. The nine middle groups were connected downward to non-overlapping, contiguous, 4x4(x4) regions of the input layer, and the 1(x28) output group was fully connected downward to the middle layer and upward to the 1(x1) top group. All connection weights (the free parameters) were initialized to random values subject to the normalization constraints of the algorithm. The learning rate, *L*, was typically set between .01 and .05.

During unsupervised training, after each input image was presented to the input layer, the upward and downward activations were computed and the learning law applied. The output node with the highest downward activation was considered the "winner" of each trial. A matrix histogram of

6 This relatively easy corpus was selected because the non-convergent nature of the Furf-I topology prevents it from easily learning transformational and style invariance. Although the path to overcoming these limitations seems clear, the behavior of the existing architecture has been sufficiently interesting as to supersede that pursuit for the moment. See *Expanding the Architecture* in the *Future Work* section of this paper.

7 Given the top layer's relative lack of prominence, one might wish to call this a *three* layer network.

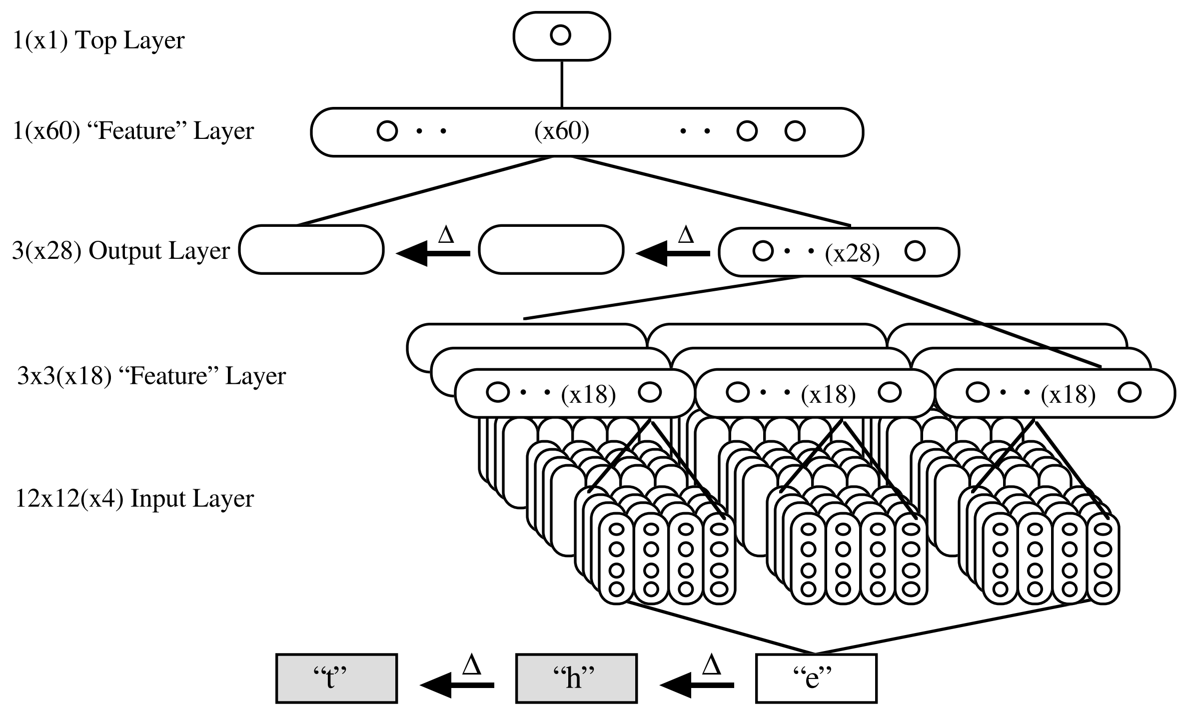
winner vs. actual letter identity was maintained and used to assign optimal labels to the output nodes for final accuracy assessment.

During supervised training, the output nodes were pre-labeled and their *downward* activations forced to represent the identity of the presented letter (one node's downward activation was forced to 1, while the rest were forced to 0). Since the downward activations were also needed for accuracy statistics, the network was run twice on each pattern—once to asses the network's guess, and once to supervise and train.

Unsupervised training produced a classification accuracy of 87%—in some trials reaching 65% before the end of the first epoch. Supervised training reached nearly 94% within the first epoch, eventually leveling out at 96%. Human performance on this data set was approximately 95% (this relatively informal measurement used one untrained subject, who had not studied this particular corpus beforehand).

Learning Context:

To study contextual learning, the previous topology was enhanced with two time-delayed copies of the output group, and a new fourth layer of 60 nodes between the now 3(x28) output layer and the top group:



This diagram illustrates the topology of the network used to study contextual learning. The input layer's groups are shown as columns to emphasize that each pixel of the letter image is represented by an entire group of four nodes. The Δ symbol represents a delay of one timestep (one pattern presentation cycle). With each cycle, the output group's *upward* activations are copied along the Δ transitions, creating a "memory" of past letters' classifications.

All training of this network was unsupervised. Training with the delays disabled produced the expected 87% (identical to the results obtained by the simpler architecture). Enabling the delays increased the final accuracy to over 91%. After training, disabling the delays decreased accuracy by only a couple tenths of a percent—producing a final single-letter recognition accuracy of nearly 91%. This represents a 30% reduction in error over the simpler architecture—despite the fact that, with delays disabled, the two networks are essentially identical. It appears that the upper feature layer was able to learn bi- and tri-gram statistics, and to use this information to self-supervise the lower layers of the network.

Pruning:

The nature of the learning law is that unused connections tend to go to zero. After training, the network was pruned by removing any node-to-node connections with probabilities near or at zero. Approximately half of the connections were eliminated, with no notable impact on performance.

Pattern Completion:

To test pattern completion, the code was modified to accept a textual representation of the image, so that various types of noise could be added by hand. Also, the input image size was increased for better visibility. No changes were made to the algorithm itself, since pattern completion is a natural consequence of the evaluation law. The Furf network reconstructed a set of degraded patterns remarkably well (although, the test patterns *were* taken from among the training set). The following is a typical example of reconstruction; the question marks represent ambiguous inputs:

```
        ###                    ###
 ??????????????               #####
 ?     ## .#.                 ## .#.
 ?+#+ ##   ##             +#+ ##   ##
 ??## ##   ##              ## ##   ##
  ????????  #              ## #     #
   +#+#  ?  #              +#+#     #
   +###  ?????             +###     #
   +###     #?             +###     #
??????????  #?             ###      #
   +##.  ?  #?             +##.     #
   +##   ?  #?             +##      #
   +##   ? .#?             +##     .#
   +#????? ##?             +#      ##
   +#?     ##?             +#      ##
   +#?????????             +##
   +#                      +#
   +#                      +#
   .#                      .#
```

Conclusion

Approaching learning from the perspective of *knowledge acquisition* has many advantages. In particular, by eliminating the notion of *output* we avoid a number of associated difficulties. Most simply, the problem of assigning *meaning* to the output is circumvented altogether. Similarly, there is no need to supply examples of desired output, or to supervise the learning process in any other manner. Learning laws which focus on output often neglect needs beyond that focus—such as the prerequisite knowledge required in supervised learning, or the global coordination lacking in most unsupervised algorithms. A knowledge-acquiring system, on the other hand, has guidance from beginning to end—from the simple absorption of fundamental relationships, to the global coordination of the process of abstraction.

Modeling efficacy is a simple, objective measure of a system's knowledge. Conceptually, it allows us to view and design the system as a pattern generator, which is considerably easier than trying to directly visualize the process of abstraction. Mathematically, it provides a concrete goal for gradient ascent.

Furf-I is an implementation of these principles. The empirical studies demonstrate rapid, unsupervised learning of real-world data, and give evidence of self-supervised, globally-coordinated abstraction. For a first attempt at implementing these principles, the empirical results are very encouraging.

Future Work

Expanding the Architecture

The current Furf-I architecture is severely constrained in that a node may only receive downward connections from a single group. The consequence of this is that when multiple independent features are present in the same input region, each possible combination of those features must be uniquely represented in the target group. This combinatorial explosion of features can be avoided if each node can be controlled from above by multiple groups. Techniques for implementing this re-convergence are presently being explored.

By introducing downward convergence to the topology, a number of possibilities would arise. For instance, connections could span multiple layers—allowing, at the extreme, fully interconnected networks. Recurrent topologies would also benefit from downward convergence, as discussed in the following section.

Temporal Data

The general learning principles that apply to static input apply equally well to temporal (time-varying) input. Creating a temporal model requires the same steps outlined in this paper: devise a model architecture capable of generating temporal patterns, then derive the gradient of the probability that that model would have generated the training set.

The crucial step here lies in devising the original architecture. For example, consider a Furf-I network consisting of two groups, A and B. Define B as the input/output group, and connect it upward to A. Then connect A recurrently to itself. This configuration is analogous to a node-labeled Hidden Markov Model. Pattern generation using this topology would proceed as follows: at each time step, based on the currently selected node in A, a node would be selected in B and a new node would be selected in A (due to the recurrent connection). This process can be repeated indefinitely—with the time-varying state of B defining the temporal output of the model.

With a little thought, however, it is apparent that as a generator this model is very limited: at any point in time, only a *single* node (within A) reflects the past state of the model. This means that if we wish to represent more than one temporal process occurring simultaneously, we must enumerate in A all possible state combinations of those processes (consider, for example, the pitch, volume, and timbre of speech, which occur simultaneously and yet are not directly correspondent with each other). This results in a combinatorial explosion in the size of the model, and hence in the number of free parameters which must somehow be determined. *If it is difficult or impossible for a model to generate a pattern, it will be difficult or impossible for the model to learn that pattern.* Thus the solution lies not in finding a better learning algorithm, but first in finding a better architecture.

By adding downward convergence (as described above in *Expanding the Architecture*) to recurrent topologies, we open the door to arbitrary connectivity. As a pattern generator, such a network could maintain multiple interacting temporal processes. Although conceiving of a learning law for such a network seems somewhat confounding, we have, at least, the concrete fundamentals of pattern generation to start from.

Small Training Sets

Given any finite number of examples, there is a single "best fit" model for the world they represent. Consider a single coin which is biased to turn up heads with an unknown probability P(H). If one flip of the coin is observed, and it turns up heads, what is the probability that the same coin will turn up heads again? The answer is, remarkably, exactly 2/3. That is, over many trials with many randomly selected coins, two thirds of the coins which turn up heads the first time will turn up heads the second. (Obviously in any single trial, the correct answer is P(H), but that value is unknown.) In the more general form, the “best fit” choice of P(H), having witnessed H heads and T tails from a single random coin, is:

$$P(H) = \frac{H + 1}{H + T + 2}$$

This is a very simple learning algorithm which gives optimal results for any amount of training data, no matter how large or small. Conveniently missing is any mention of a learning rate. Note also that this ideal solution differs from the solution a simple gradient ascent would find, namely H/H+T. As the number of samples (H+T) grows, the two formulae converge to the same answer, but for very small training sets the answers can be quite different. It may be possible using this type of analysis to eliminate, or at least automatically and dynamically determine, the learning rate used in the Furf learning law.

Acknowledgements

I would like to thank Josh Gold, Michael Kaplan, and Larry Yaeger for their time and patience in discussing these ideas with me, and for not being afraid to tell me from time to time that I was insane. I would also like to thank Phil Steele for his invaluable contributions toward general clarity and for finding most of the spelling errors that the computer mised.

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